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A CONSTRUCTION FOR THE IMAGINARY POINTS AND BRANCHES OF PLANE CURVES.

By Prof. F. H. Loud, Colorado Springs, Col.

I. Historical Introduction.

1. "In the investigation of the algebraic function y of a variable x we are wont to employ two different means of geometric representation [anschauungsmässiger Hülfsmittel]. We either represent x and y alike as co-ordinates of a point of the plane,—where then their real values alone come into evidence, and the image of the algebraic function is the algebraic curve,—or we spread out over a plane the complex values of one variable, x, and express the functional relation between y and x by the Riemann's surfaces constructed on the plane. It must in many cases be desirable to possess a transition between these two geometric images."

When it is stated that the foregoing is a translation of the introductory sentences of an article by Felix Klein,* no further vindication will be needed of the intrinsic interest of the subject-matter of the present essay. Rather the question will be at once suggested, whether anything can remain to be said upon the topic, after the elegant solution which this prince of modern mathematicians has of course given for the problem he has thus broached. And still more questionable does such a possibility appear, when it is further stated that the method proposed in the article cited has since received a material addition from the hand of the same master.

The element of novelty in the present attempt, however, does not descend to the fundamental principles of the geometric representation of imaginaries,—were this the case, the proposed method must be so unrelated to modern work as to lose a main element of possible usefulness,—but while wrought out with a view to an application to problems as elementary as those are profound which prompted the investigation of Klein, it appears after having been thus independently reached to be capable of statement as a modification or specialization,—with slight addition,—of the principles stated by him.

2. *J. V. Poncelet,* 1822. The problem, to represent graphically the imaginary elements of plane curves, has indeed long engaged attention. The generality of view attained by Poncelet in his ever memorable "*Traité des*

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* Ueber eine neue Art der Riemann'schen Flächen, Mathematische Annalen, Bd. VII.

Propriétés Projectives des Figures" was largely due to his being able to embrace under one statement the properties of real and ideal secants;—i. e. of lines meeting a curve in real or imaginary points. If OM, the half of a real chord parallel to a given ideal secant, were found to have a length determined by the equation $\overline{OM}^2 = p \,.\, AO \,.\, OB$, (where A and B are the vertices of the bisecting diameter) then $O'M'$, determined by $\overline{O'M'}^2 = p \,.\, O'A \,.\, O'B$, and measured both ways on the ideal secant from O', its intersection with the same diameter, was an "ideal chord;" and the real point M' is evidently regarded as the image of the imaginary intersection of the line and curve. The locus of M', named by Poncelet a "courbe supplementaire," is a conic of opposite species to the given one, and tangent to it at the vertices of the diameter; and, as the number of these curves is infinite, the doubly infinite aggregate of imaginary points belonging to the curve is completely represented by a device which, it is apparent, amounts simply to the substitution, under definite conditions, of y for $y\sqrt{-1}$.

3. *C. F. Maximilien Marie*, 1842 *and* 1874. The method of Marie may be regarded as an extension to curves in general, and to surfaces, of that employed by Poncelet for real conics. It is presented in its final form in a work issued in three parts in the years 1874, 1875, and 1876 under the title "*Theorie des Fonctions de Variables Imaginaires;*" but while these comparatively recent dates are affixed to the volume, of which the author says "*L'histoire de ce livre est en quelque sorte mon histoire*," he states that a paper containing the notion of "conjugates, such as I have defined them in this work," was communicated to the Academy April 15th, 1842.

If $x = \alpha + \beta\sqrt{-1}$, $y = \alpha' + \beta'\sqrt{-1}$ be a set of values satisfying the equation $F(x, y) = 0$, the author shows that this, together with all the singly infinite series of points for which the ratio of the imaginary terms is equal to β'/β, may be made by a rotation of the axis of Y to have a real abscissa, the transformed co-ordinates being x', and $y' = \varphi(x') + \sqrt{-1}\,\psi(x')$. In place of the latter, he substitutes $\varphi(x') + \psi(x')$ and constructs with these real co-ordinates a point of the "conjugate" curve characterized by the ratio $\beta'/\beta = C$. Each new value of C requires a different rotation of the axis and defines a new conjugate. If the locus of $F(x, y)$ have a real branch, this (as with Poncelet) appears not as one of the series of [supplementary or] conjugate curves, but as their envelope. But loci entirely imaginary may be represented on the same plan. To obtain in real co-ordinates the equation of a conjugate, Marie substitutes in $F(x, y) = 0$ the values $\alpha + \beta\sqrt{-1}$ for x and $\alpha' + \beta C\sqrt{-1}$ for y; then, writing the real and imaginary terms separately, he has the two equations $f(\alpha, \beta, \alpha') = 0$ and $f_1(\alpha, \beta, \alpha') = 0$; and between these, with the aid of the equations $x_1 = \alpha + \beta$, $y_1 = \alpha' + \beta C$, he eliminates α, β, and α'. The result will in general be of the degree n^2 (that of F having been n), but if the coefficients of F are all real, the degree is $n(n-1)$. Hence the conjugates of a real conic are conics, and indeed,—as already intimated,—are identical with Poncelet's "supplementaires." Those of a real right line coincide in that line. This introduces a valuable simplicity in linear constructions; and the method has evidently some excellent qualities, coupled with what,—in view of the fruitful employment by Poncelet of so similar a device,—one hesitates to call an unscientific feature; viz., the fact that the representation of imaginaries is secured by merely throwing away the imaginary factor from the terms in which it occurs. This feature has not been excluded from some writings of more recent date; thus a tract entitled "*Imaginäre Kegelschnitte: eine geometrische Studie über das Wesen und die katoprische Deutung des Imaginären;* von Adalbert Breuer, Erfurt, 1892," may be cited as belonging to the school of Poncelet and Marie.

4. *J. R. Argand*, 1806 = *C. F. Gauss*, 1831. That representation of imaginaries, however, which was destined to win general acceptance had already been proposed by Argand and introduced anew to the mathematical world by Gauss. From the metrical point of view, it is so simply associated with that natural conception of negative quantities which lies at the basis of the Cartesian geometry, that the element of arbitrariness may be said to disappear. For the geometrical expression of a single finite imaginary, $m + i\mu$, it seems perfect. But how to apply it to represent a function of a complex variable is not immediately evident. As two rectangular axes on which m and μ are independently laid off, are required to construct $x = m + i\mu$, it would appear that two more would be needed for $y = n + i\nu$. Thus the imaginary points of a real plane require a construction in four dimensions and although those of a locus $f(x, y) = 0$ are extended in only two, this surface cannot be directly brought within the realm of intuitive apprehension. It might, indeed, be represented by its projection from an arbitrary point on a plane space of three dimensions. Again, if only one of the variables be allowed to take complex values, there is no difficulty in constructing the locus in tridimensional space, as is done in Phillips and Beebe's "Graphic Algebra," but this condition is not sufficiently general for analytical geometry. In either of these two cases it is clear that the representative capacity of three-dimensioned space is exhausted in providing for the imaginaries of the "plane locus" $f(x, y) = 0$, while those of the surface $f(x, y, z) = 0$ must remain impossible quantities. A construction capable of extension, like Marie's, to three variables, must, like his, represent the double infinity of imaginary points of a plane by an infinite series of series of elements, all situated in the plane. On this account it will be unnecessary to devote any space to Riemann's method of representing imag-

inaries by the points of a *spherical* surface,—a representation which, by means of a stereographic projection, conducts back to that of Argand.

5. *K. G. C. von Staudt*, 1847. The foregoing requirement, together with the demand of strict scientific accuracy, was met by von Staudt's construction given in the "*Beiträge zur Geometrie der Lage;*" which employs as the image of an imaginary point an involution upon the real line connecting the point in question with its conjugate; and, since the two conjugate points are thus represented by the same involution, distinguishes between them by combining therewith a determinate direction along the line. It is not difficult to detect a close relationship between this representation and that of Poncelet; while on the other hand, a new statement of Argand's method is given,* which reduces the latter to a special case of a projective process. The imaginary number $m + i\mu$ may be regarded as a parameter defining one of a double infinity of lines drawn through a fixed point, namely, through that one of the two "circular points at infinity" which lies on the line $x + iy =$ *constant*, and which we may distinguish from the conjugate circular point (on $x - iy =$ *constant*) by naming the former K_1, the latter K_2. Each line of this pencil,—for example, $x + iy = m + i\mu$,—will be met in a real point by that line of a second pencil, having K_2 as centre, which is characterized by the conjugate number $m - i\mu$; and this real point is (m, μ) the same by which Argand represents the given complex quantity.

6. *B. Riemann*, 1859. The founder of the modern theory of the functions of an imaginary variable applied the construction of Argand to represent the x and the y of $f(x, y) = 0$ each in a separate plane, between which planes no geometrical relation is assumed. In either, a point may be moved at will, when the resulting motion of the correlated point in the other will serve as a geometric expression of the function, quite unrelated, of course, to Descartes' plane curve $f(x, y) = 0$.

7. *F. Klein*, 1874. In the paper quoted at the beginning of the present article, Professor Klein proposes to consider a plane curve as the envelope of its tangents, and to co-ordinate with each ordinary† tangent one real point; namely, if the tangent be real, the point of contact; if it be an imaginary line, the real point of the latter. If now we regard these real points as images of

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* A brief statement of the projective relations of the various methods of representation of imaginaries may be found in the introductory paragraphs of the paper of Corrado Ségré "*Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici.*" (*Math. Ann.*, Bd. XL.) For a recapitulation of the diverse philosophic theories of the nature of $\sqrt{-1}$, reference may be made to the paper "*On the Imaginary of Algebra*" read before the A. A. A. S. at its forty-first meeting by Prof. A. Macfarlane, who gives due credit to Bueé and other contemporaries of Argand.

† Double and inflectional tangents are separately considered, but are for brevity omitted from the present sketch.

the real or imaginary points of contact of the several tangents, all the real points of the curve are their own images, while every imaginary point is imaged in a real point, not lying on the curve. The latter points, though doubly infinite in number, are not necessarily distributed over the whole plane, but cover those portions of it from which imaginary tangents can be drawn; and the number of such tangents from each real point of any portion of the plane, (always even if the curve is real) shows how many times that portion is covered over by the surface composed of these images. These are the "new kind of Riemann's surfaces," and are employed by the author to determine directly the deficiency (*Geschlecht*) of the curve. This use,—and in general the treatment of problems relating to integrals extended along the curve, their periodicity, etc.,—are the applications suggested by him, and it does not appear that for such purposes, any problems relating to the intersections of curves require consideration. In elementary geometry such problems of course constantly occur, and the difficulties that would arise in applying this method to them may be inferred from the fact that an imaginary intersection would be represented by different real points according as it were regarded as belonging to one curve or to the other.

8. *F. Klein*, 1892. Later, Professor Klein has proposed a second way of representing the imaginary points of a curve, which he distinguishes as a "metrical" method from the foregoing "projective" one. The metrical plan is described in his "*Vorlesungen über Riemann'sche Flächen*," (Wintersemester, 1891–2; pp. 218 *et seq.*), but whether some reference to it is to be found in any of his previous works I am not able to say. From any real or imaginary point, P, of the curve a line is drawn to the circular point K_1, and the intersection of this line by that which joins K_2 to the point conjugate to P,—in short, the real point of PK_1,—is the image of the point P. The resulting representation differs in important respects from the foregoing. A point of given co-ordinates is always represented by the same real point of the plane, independently of any curves on which it may be situated. While the former method is remotely, the latter is closely related to Argand's scheme of representation, and hence admits of being stated,—like Argand's,—from an elementary metrical point of view. It is, in fact, identical with that part of the scheme proposed in the second division of the present paper which is comprised in paragraph 11. While by the *projective* method, the plane is divided up,—as regards the number of times it is covered over by the images of the imaginary points,—into different regions, unequally covered, and so that the real branches of the curve, when they exist, are always boundaries between different regions, by the *metrical* plan the entire plane, bounded by the line infinity, is covered alike; and the number of times, instead of depending on the class or

deficiency of the curve, is equal to its order diminished by the number of times it passes through the point K_1. While the foci of the curve are, as Professor Klein points out, "*Verzweigungspunkte*" of the surface, the ordinary points lying on the curve have no special distinction. Hence while the projective method offers directly, by bringing the real branches into prominence, that transition to the Cartesian construction which its author, (in the paragraph quoted at the beginning of this article,) demanded of it, it would seem necessary, in using the metrical method for the same purpose, to adopt some means of indicating that law of distribution of the complex points over the plane which, in the special case of the vanishing of the imaginary terms, results in locating the real branches. Professor Klein, in applying the projective method to the discussion of integrals, finds it convenient to draw a system of "meridians and curves of latitude," the real branches of the curve falling into place as members of the latter series. Can a similar device be attached to the metrical method? This question is akin to one discussed in the lectures cited in regard to the relation which the surfaces presented by the n-times-covered plane of this method sustain to the ordinary Riemann's surfaces. It is shown that if we assume

$$z = x + iy\,, \quad s = x - iy\,, \quad F(s, z) = f(x, y)\,,$$

then a point x, y,—these co-ordinates being in general complex,—is represented in the z-plane at the same position to which it is assigned by the metrical representation. Since, then, we may produce in this z-plane an infinite variety of curves, by proper choice of a path for a point in the plane of s, taken as an independent variable, it is natural to inquire for the path whose image in the z-plane shall be the real curve. The equations just written indicate that, x and y being in this case real, the curves in the z-plane and s-plane will be equal, and symmetrically placed in reference to the axes of reals. That is, in order to produce the desired Cartesian curve in the z-plane, its form must be previously known and assumed in the s-plane.

9. *Method of the present paper.* But if, dismissing the variable s, we obtain from the equations $f(x, y) = 0$ and $z = x + iy$, by eliminating first y and then x, two new equations $\varphi(x, z) = 0$ and $\psi(y, z) = 0$, we may obtain in the z-plane a curve consisting, if not altogether at least in part, of the real Cartesian curve, by simply moving an independently variable point in either the x-plane or the y-plane *along the axis of reals*. The images in the z-plane of these right lines and of parallels to them are precisely the curves defined as "comitants" in the second division of the present paper. As I cannot find that this particular modification of Professor Klein's method has been pro-

posed by another, it may be that for these "comitants,"—however much or little they may be worth,—I am entitled to make the claim of novelty.

II. THE PLAN OF REPRESENTATION HEREIN PROPOSED.

10. *Object.* As already intimated, this plan was devised without the knowledge of the above described work of Professor Klein, and for a purpose very remote from such researches at the advancing border of human knowledge as he is wont to conduct. The original aim was pedagogic,—to present the imaginaries of analytical geometry to the elementary student in a concrete form, so that they should be freed from all suspicion of *unreality*, and by a method so related to the usual Cartesian construction that the geometric forms and processes of the latter should come out,—as their algebraic equivalents clearly are,—as mere special cases characterized by the vanishing of the imaginary terms. Such constructions must be more complicated than those of Descartes, as the imaginary points of a locus are infinitely more numerous than the real ones; and it is not expected that the student will carry the method into every investigation; but it is believed that a great advantage will be gained if he is aware of the existence of such a scheme of representation, understands its principles, and is able to resort to it when occasion requires. Should the device be successful in the elementary field, it may then be hoped,—and here its relationship to the methods of the best authors is encouraging,—that it may be applied with advantage in the theory of functions. Trigonometrical functions of a complex variable, in particular, ought to be capable of being studied in close relationship with the ordinary geometrical definitions of the like functions of a real angle. Finally, the series of comitants afford instances of systems of curves under special relations, which may present interesting features. But the first object named appears to me fundamental, and hence I take the liberty of calling attention to the two following paragraphs, (11 and 12) as indicating, in the unbracketed parts, the order in which the plan is intended to be presented to a student, supposed already familiar with the construction of Argand. It is to be premised that the rectangular axes assumed are an axis of reals and an axis of pure imaginaries,—*not axes of X and Y.*

11. *Complex Points.* The position denoted by $x + iy$ is that at which Descartes places the point whose co-ordinates are x and y, when these quantities are real. We will follow the same plan if they are complex. Thus the point $x + i\xi, y + i\eta$ will occupy the same position as the real point $x - \eta, y + \xi$. If the co-ordinates of a point are infinite they may or may not satisfy the equation $x + iy = finite$, and will accordingly be situated at a finite or an in-

finite distance from the origin. [This is as much as to say that the position of K_1 is indeterminate in the finite region, while all other points having infinite co-ordinates are at infinity. Compare paragraph 8 *ante*.] A set of co-ordinates satisfying an equation $f(x, y) = 0$ of the nth degree defines a single point of the locus, but n of these points fall on every point of the plane ; [unless K_1 is a p-fold point of the locus, when this number is reduced to $n - p$].

12. *Comitants.* A curve drawn through those points of the locus whose abscissas have in common the imaginary term $i\mu$ is named the *x-comitant* μ, and μ is the *characteristic* of the comitant. As μ takes in succession an infinity of different values, a series of x-comitants is obtained ; and if in the above definition " abscissa " be replaced by " ordinate " there is derived a corresponding series of y-comitants. If the locus have a real branch, this is a common arc of the two comitants zero.

When two loci intersect,—say in the point $m + i\mu$, $n + i\nu$, this is on the x-comitant μ of each, and is also common to their y-comitants ν. So also a point of contact of two loci is that at which a comitant of *each* series belonging to one locus is touched by the comitant of equal characteristic in the like series of the other locus. If the point of contact is at infinity, μ and ν are indeterminate ; accordingly each comitant of either series in one locus has for an asymptote the corresponding comitant of the other. [One of the circular asymptotes, as above noted, is exceptional.]

A rule for the construction of comitants by points, when the constants are given either numerically or geometrically, is obtained by inspection of the equation of the locus, regard being had to the laws of geometric addition and proportion ; and when the locus is not above the second order, can involve only processes of elementary geometry. In particular, the comitants of the general locus of first order are right lines, all those of either series being parallel, while the two series make with each other an angle equal to the argument of the ratio between the coefficients of the two variable terms. So when this ratio is real, all the comitants of both series are parallel.

13. *Order of comitants.* The Cartesian equation of a comitant may be obtained by the following rule : After separating the real and imaginary terms of $f(x, y) = 0$, substitute in the two resulting equations $x + \eta$ for x and $y - \xi$ for y (see paragraph 11) and eliminate either η or ξ according as an equation for x-comitants or for y-comitants is desired. The result, as was observed of Marie's " conjugates," is, at highest, of the degree n^2. This degree is depressed not, as in that case, for real loci, but for those containing the circular point K_1. Thus the comitants of the circle are cubics.* The two comitant-equations

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* A discussion of this case is given in my article " On the Circular Locus " (Colorado College Studies, Fourth Annual Publication).

derived from the general equation of first degree,

$$(a + i\alpha)x + (b + i\beta)y + c + i\gamma = 0$$

are

$$(ab + \alpha\beta)x + (b^2 + \beta^2 - a\beta + b\alpha)y - [(a - \beta)^2 + (b + \alpha)^2]\xi$$
$$+ (bc + \beta\gamma - a\gamma + c\alpha) = 0\,,$$

and

$$(a^2 + \alpha^2 - a\beta + b\alpha)x + (ab + \alpha\beta)y + [(a - \beta)^2 + (b + \alpha)^2]\eta$$
$$+ (ac + \alpha\gamma + b\gamma - c\beta) = 0\,.$$

Colorado College, *Oct.* 3, 1893.